

BURNSIDE PROBLEM FOR GROUPS OF HOMEOMORPHISMS OF COMPACT SURFACES.

NANCY GUELMAN AND ISABELLE LIOUSSE

ABSTRACT. A group Γ is said to be periodic if for any g in Γ there is a positive integer n with $g^n = id$. We first prove that a finitely generated periodic group acting on the 2-sphere \mathbb{S}^2 by C^1 -diffeomorphisms with a finite orbit, is finite and conjugate to a subgroup of $O(3, \mathbb{R})$ and we use it for proving that a finitely generated periodic group of spherical diffeomorphisms with even bounded orders is finite.

Finally, we show that a finitely generated periodic group of homeomorphisms of any orientable compact surface other than the 2-sphere or the 2-torus (which is the purpose of a previous paper of the authors) is finite.

1. INTRODUCTION.

Definition 1.1. A group Γ is said to be **periodic** if any g in Γ has finite order, that is, there exists a positive integer n with $g^n = id$.

One of the oldest problem in group theory was first posed by William Burnside in 1902 (see [2]): “Let Γ be a finitely generated periodic group. Is Γ necessary a finite group?”

It is obvious that an abelian finitely generated periodic group is finite.

In 1911, Schur (see [18]) proved that this is true for subgroups of $GL(k, \mathbb{C})$, $k \in \mathbb{N}$.

But, in general, according to Golod (see [7]) the answer is negative. Later, Adjan and Novikov (see [1]), Ol’shanskii, Ivanov and Lysenok (see [15], [9] and [12]) exhibited many examples of infinite, finitely generated and periodic groups with even bounded orders.

The problem raised by Burnside is still open for groups of homeomorphisms (or diffeomorphisms) on closed manifolds. Very few examples are known.

As a corollary of Hölder theorem (see, for example, section 2.2.4 of [14]), it holds that a finitely generated periodic group of circle homeomorphisms is finite. Note that, even in this case, finiteness of the generating set is crucial: the group consisting of all rational circle rotations is periodic and infinite.

Rebello and Silva (see [16]) proved that any finitely generated periodic subgroup of C^2 -symplectomorphisms of a compact 4-dimensional symplectic manifold is finite, provided that the fundamental class in $H^4(M, \mathbb{Z})$ is a product of classes in $H^1(M, \mathbb{Z})$.

The authors proved (see [8]) that any finitely generated periodic subgroup of $\text{Homeo}_\mu(\mathbb{T}^2)$ is finite, where μ is a probability measure on \mathbb{T}^2 and $\text{Homeo}_\mu(\mathbb{T}^2)$ is the subgroup of orientation-preserving homeomorphisms of \mathbb{T}^2 preserving μ . Moreover, they showed that every finitely generated 2-group of toral homeomorphisms is finite.

As a consequence of this result, we get the following

Corollary 1. Any finitely generated periodic subgroup of $\text{Homeo}(\mathbb{A}^2)$ is finite, where \mathbb{A}^2 is the closed annulus.

This paper was partially supported by the Labex CEMPI (ANR-11-LABX-0007-01), Université de Lille 1, PEDECIBA, Universidad de la República and IFUM.

The idea of the proof is to form the double of \mathbb{A}^2 by gluing two copies of \mathbb{A}^2 along their boundaries. This will be explained in section 5.

In this paper, we study related questions. We first consider finitely generated periodic groups of diffeomorphisms of the 2-sphere, \mathbb{S}^2 and prove

Theorem 1.

Let Γ be a finitely generated periodic group of C^1 -diffeomorphisms of \mathbb{S}^2 . If Γ has a finite orbit then it is finite.

Moreover if Γ consists in orientation preserving C^1 -diffeomorphisms and acts with a global fixed point then it is abelian.

In order to relax the differentiability hypothesis, we have as a consequence of the proof of Proposition 4.3, the following

Corollary 2. *If G is a finitely generated periodic group of orientation preserving homeomorphisms of \mathbb{S}^2 and G has a finite orbit of cardinality at least 3, then it is finite.*

In some cases, we are able to establish the existence of a finite orbit, in particular we have the following

Corollary 3. *If G is a finitely generated periodic group of spherical diffeomorphisms with even bounded orders then G is finite.*

Let Γ be a finite group, by a classical result of Kerekjarto and Eilenberg (see [3],[4], [5], [10]), every topological action of Γ on the 2-sphere is conjugate to an orthogonal action (i.e by orthogonal maps of $O(3, \mathbb{R})$).

Moreover, as it is explained in [13], if the finite group Γ acts smoothly on a closed surface, Γ leaves invariant some Riemannian metric of constant curvature. Thus any action of Γ on the 2-sphere is conjugate in $\text{Diff}^1(\mathbb{S}^2)$ to an orthogonal action. We will refer to this result as the “folkloric” one.

Combining these results with Theorem 1 and Corollary 2 we have the following rigidity result.

Corollary 4. *Let Γ be a finitely generated periodic group.*

1. If Γ acts by C^1 -diffeomorphisms of \mathbb{S}^2 with a finite orbit then it is conjugate in $\text{Diff}^1(\mathbb{S}^2)$ to a finite orthogonal group, that is a finite subgroup of $O(3, \mathbb{R})$.

2. If Γ acts by orientation preserving homeomorphisms of \mathbb{S}^2 with a finite orbit of cardinality at least 3, then it is conjugate to a finite orthogonal group.

We note that the only “interesting compact surfaces” for studying Burnside Problem are the sphere and the torus. In section 5 we will prove

Theorem 2. *Any finitely generated periodic group acting by homeomorphisms on a compact orientable surface S of genus $g \geq 2$, is a finite group.*

As it can be verified in Section 5, this kind of results for orientable compact surfaces are consequences of well known results. One of them (see [5]) is that any periodic homeomorphism in any compact surface preserves a Riemannian metric of constant curvature. By Killing-Hopf Theorem (see, for example section 6.2 of [17]) the curvature is -1 if the genus of the surface is greater than one and it is 0 for the torus and it is 1 for the sphere.

As Katherine Mann showed us, the proof of Theorem 1 can be extended to any manifold, so we have the following

Theorem 3.

Let M be a compact riemannian manifold and let Γ be a finitely generated periodic group of C^1 -diffeomorphisms of M . If Γ acts on M with a finite orbit or preserves a finite union of circles then Γ is finite.

As consequence we get

Corollary 5.

Let Γ be a finitely generated periodic group of orientation preserving C^1 -diffeomorphisms of \mathbb{S}^2 . If Γ has a non trivial center $Z(\Gamma)$ then Γ is finite.

Theorem 1 will be proved in sections 2, 3, 4 and section 5 is devoted to orientable compact surfaces, in particular this section contains the proofs of Corollary 1 and Theorem 2. Theorem 3 and Corollary 5 will be proved in section 6. Corollary 3 will be proved in section 7.

Acknowledgements. We are grateful to Andrés Navas for proposing to us this subject and for fruitful discussions. We thank K. Parwani for suggesting us the Burnside problem on the closed annulus and the closed disk. We also thanks to F. Leroux, J. Franks and K. Mann for several useful suggestions.

2. CLASSIFICATION OF FINITE ORDER ORIENTATION PRESERVING SPHERICAL HOMEOMORPHISMS.

In this section, we recall the classification of finite order orientation preserving spherical homeomorphisms of sphere up to conjugacy.

According to [3], [4], [5] or [10], a finite order spherical homeomorphism is conjugate to an orthogonal map of $O(3, \mathbb{R})$ and using the classification of elements in the orthogonal group $O(3, \mathbb{R})$, we get the following definition and proposition.

Definition 2.1. *A spherical homeomorphism is called **quasi-rotation** if it is conjugated to a spherical rotation.*

Proposition 2.1. *A finite order, orientation preserving spherical homeomorphism is a quasi-rotation and then if it is non trivial it has exactly two fixed points.*

Remark 1. *The “folkloric result” states that if the homeomorphism is a C^1 -diffeomorphism then the conjugating map is also a C^1 -diffeomorphism.*

As a corollary of Proposition 2.1, we get

Corollary 6. *If Γ is a periodic group, then \mathbf{G} the subgroup $\{g \in \Gamma : g \text{ is orientation preserving}\}$ is exactly the set $\{g \in \Gamma : g \text{ is a quasi-rotation}\}$. In particular, the subset of Γ consisting in quasi-rotations is a subgroup.*

3. REDUCTION TO GROUPS OF RATIONAL QUASI-ROTATIONS.

A direct consequence of the fact that the composition of two orientation reversing homeomorphisms is orientation preserving, is

Lemma 3.1. *Let Γ be a finitely generated periodic group of spherical homeomorphisms. Then G , the subgroup of Γ consisting in quasi-rotations is of finite index in Γ . Hence, Γ is finite if and only if G is finite.*

For proving Theorem 1, it is enough to prove that G is finite. This is the purpose of the following section. Our proof consists in considering the following three cases: the case where G has a global fixed point achieved in Proposition 4.1, the case where G has a finite orbit of cardinality 2 proven in Proposition 4.2 and the remaining case where G has a finite orbit of cardinality at least 3 showed in Proposition 4.3.

4. BURNSIDE PROBLEM FOR GROUPS OF RATIONAL QUASI-ROTATIONS.

Let G be a finitely generated periodic group of quasi-rotations of the sphere.

Definition 4.1. We denote by \mathbf{P}_G the set of points in \mathbb{S}^2 that arise as fixed points of some non trivial element of G .

Let $x \in P_G$, we denote by $\mathbf{St}_G(x)$ the **stabilizer** in G of x , that is the set:

$$\mathbf{St}_G(x) = \{g \in G : g(x) = x\}.$$

Lemma 4.1. The set P_G is G -invariant and $G = \bigcup_{x \in P_G} \mathbf{St}_G(x)$.

Proof. Let $x_0 \in P_G$ and $g \in G$. By definition, there exists $f \in G$ such that $f(x_0) = x_0$. As $g \circ f \circ g^{-1}(g(x_0)) = g \circ f(x_0) = g(x_0)$, it follows that $g(x_0) \in P_G$.

The second point is a direct consequence of the fact that any element in G admits a fixed point (every non trivial quasi-rotation has exactly two fixed points). \square

4.1. Groups acting with a global fixed point.

Definition 4.2. Let $x \in \mathbb{S}^2$, we define the following groups:

- $\mathbf{Diff}_+^1(\mathbb{S}^2)$ consisting in orientation preserving C^1 spherical diffeomorphisms,
- $\mathbf{Diff}_x^1(\mathbb{S}^2)$ consisting in C^1 spherical diffeomorphisms fixing x and
- $\mathbf{Diff}_{x,+}^1(\mathbb{S}^2)$ their intersection subgroup.

This section contains results on finitely generated periodic subgroups of $\mathbf{Diff}_{x,+}^1(\mathbb{S}^2)$. Furthermore, in the next section, we will apply these results in order to describe stabilizers of points that might not be finitely generated.

Definition 4.3. Define the map $\mathbf{D} : \mathbf{Diff}_x^1(\mathbb{S}^2) \rightarrow \mathrm{GL}(2, \mathbb{R})$ by

$$D(g) = Dg(x) : \mathbb{R}^2 \approx T_x \mathbb{S}^2 \rightarrow \mathbb{R}^2 \approx T_x \mathbb{S}^2, \text{ the differential map at } x.$$

Lemma 4.2. The map D is a morphism and the image of a periodic subgroup of $\mathbf{Diff}_{x,+}^1(\mathbb{S}^2)$ is a periodic subgroup of $\mathrm{SL}(2, \mathbb{R})$.

Proof. As $D(fg)(x) = Df(g(x))Dg(x) = Df(x)Dg(x)$ for any f, g in $\mathbf{Diff}_{x,+}^1(\mathbb{S}^2)$, D is a morphism. Moreover, $D(g)$ has finite order provided that g has.

Let g be a finite order element in $\mathbf{Diff}_{x,+}^1(\mathbb{S}^2)$. By Proposition 2.1 and Remark 1, there exist a spherical diffeomorphism h and a spherical rational rotation R_α such that $g = h^{-1}R_\alpha h$. Without loss of generality, we can assume that $h(x) = x$ and $R_\alpha(x) = x$.

Indeed, if not $y := h(x) \neq x$. There exists a spherical rotation R_β such that $R_\beta(y) = x$ and therefore $R_\beta h(x) = x$ and we can rewrite $g = (R_\beta h)^{-1}(R_\beta R_\alpha R_\beta^{-1})(R_\beta h)$.

Thus $g = H^{-1}R'_\alpha H$, where $H = R_\beta h$ fixes x and $R'_\alpha = R_\beta R_\alpha R_\beta^{-1}$ is a spherical rotation and $R'_\alpha(x) = HgH^{-1}(x) = Hg(x) = H(x) = x$.

Finally, we have $D(g) = D(h)^{-1}D(R_\alpha)D(h)$, since D is a morphism. The linear map $D(R_\alpha)$ is the planar rotation of angle α , then $D(g)$ has determinant equal to 1 so it belongs to $\text{SL}(2, \mathbb{R})$. \square

Proposition 4.1. *Let G be a finitely generated periodic subgroup of $\text{Diff}_{x,+}^1(\mathbb{S}^2)$. Then G is finite and abelian.*

Proof. The set $D(G)$ is a finitely generated periodic subgroup, since it is the image by a morphism of G satisfying these properties. According to Schur's theorem ([18]), as $D(G)$ is a finitely generated periodic subgroup of $\text{SL}(2, \mathbb{R})$, it is finite then it is compact.

The classification of compact subgroups of $\text{SL}(2, \mathbb{R})$ states that $D(G)$ is conjugated to a subgroup of $\text{SO}(2, \mathbb{R})$ consisting in linear planar rotations (see for example [11]), hence $D(G)$ is abelian.

As a consequence, for any f, g in G , $D([f, g]) = \text{Id}$, where $[f, g] = fgf^{-1}g^{-1}$ is the commutator of f and g .

Finally, $[f, g] = hR_\alpha h^{-1}$, since it is a quasi-rotation. Then $D([f, g]) = D(hR_\alpha h^{-1}) = D(h)D(R_\alpha)D(h)^{-1} = \text{Id}$ and therefore $D(R_\alpha) = \text{Id}$ so $\alpha = 0$ and $R_\alpha = \text{Id}$. Hence $[f, g] = \text{Id}$.

Consequently G is a periodic, finitely generated and abelian group. It follows that G is finite. \square

4.2. G has a finite orbit of cardinality 2.

An important ingredient in this case is the following lemma concerning stabilizers of points. As a consequence of Proposition 4.1, we have

Lemma 4.3. *Let $x_0 \in P_G$ and G' be a periodic subgroup of $\text{Diff}_{x_0,+}^1(\mathbb{S}^2)$, then G' is an abelian group and its elements have the same two fixed points. Moreover, any finitely generated subgroup of G' is finite and conjugated to a group of rational spherical rotations of same axis.*

In particular, if G is a finitely generated periodic subgroup of $\text{Diff}_+^1(\mathbb{S}^2)$, the subgroup $G' := \text{St}_G(x_0)$ is an abelian group.

Proof. Let f, g in G' . The group $\langle f, g \rangle$ generated by f and g is a finitely generated periodic subgroup of $\text{Diff}_+^1(\mathbb{S}^2)$ that fixes x_0 . Hence, according to Proposition 4.1, $\langle f, g \rangle$ is finite and abelian. Consequently, f and g commute.

Moreover, $fgf^{-1} = g$ implies that $\text{Fix}(g) = f(\text{Fix}(g))$. Let $y \neq x$ be the second fixed point of the quasi-rotation g . Then $\text{Fix}(g) = \{x, y\} = \{f(x) = x, f(y)\}$, then $f(y) = y$.

Any finitely generated subgroup of G' is abelian and periodic, so it is a finite subgroup of $\text{Diff}_+^1(\mathbb{S}^2)$. By the folkloric result, it is C^1 -conjugated to a subgroup of rational rotations. \square

Proposition 4.2. *If G is a finitely generated periodic subgroup of C^1 quasi-rotations. If G has a finite orbit of cardinality 2, then G is finite.*

Proof. Let x_0 be a point with G -orbit of cardinality 2. We write $\mathcal{O}_G(x_0) = \{x_0, x'_0\}$.

By Corollary 4.3, $\text{St}_G(x_0)$ is abelian.

We claim that $[G, G]$, the first derivated subgroup of G , is contained in $\text{St}_G(x_0)$.

Let $g \in G$, we have $g(x_0) \in \mathcal{O}_G(x_0) = \{x_0, x'_0\}$ and $g(x'_0) \in \mathcal{O}_G(x_0) = \{x_0, x'_0\}$. Noting that if $g(x_0) = x'_0$ then $g^{-1}(x_0) = x'_0$, it is easy to check that $[f, g](x_0) = f^{-1}g^{-1}fg(x_0) = x_0$, in all possible cases.

We conclude that $[G, G]$ is abelian, this means that $[[G, G], [G, G]]$, the second derivated group of G , is trivial.

The last part of this proof is a general fact for finitely generated groups generated by s finite order elements g_1, \dots, g_s : “any element of G can be written $g = g_1^{p_1} \dots g_s^{p_s} C$, where $C \in [G, G]$ and $p_i \geq 0$ is bounded by the order of g_i .” So the index of $[G, G]$ in G is bounded by the product of the orders of g_1, \dots, g_s . Moreover, Schreier’s lemma states that any subgroup of finite index in a finitely generated group is finitely generated. This implies that $[G, G]$ is also finitely generated.

Here, as G is a periodic group then $[G, G]$ is a finitely generated periodic group. In particular, it is generated by finite order elements. Hence, last argument shows that $[[G, G], [G, G]]$ has finite index in $[G, G]$ which has finite index in G . Finally $[[G, G], [G, G]]$ has finite index in G . But, we also have shown that $[[G, G], [G, G]]$ is trivial, so G is finite. \square

Remark 2. Under the hypotheses that G is a finitely generated periodic group of homeomorphisms, $\#P_G = 2$ and G preserves a probability measure on $\mathbb{S}^2 \setminus P_G$, analogous arguments as those developed in [8] show that G is finite and abelian.

A sketch of the proof of last Remark is the following: G acts on the open annulus $\mathbb{S}^2 \setminus P_G$ and preserves a measure. Hence, we can define the rotation map $\rho : G \rightarrow \mathbb{S}^1$; the number $\rho(g)$ coincides with the angle of a rotation conjugated to g . As in [8], one shows that ρ is a morphism. Therefore, it vanishes on commutators. Then any commutator is conjugate to a rotation of angle 0, so it is trivial. It follows that G is abelian and since it is also finitely generated and periodic, it is finite.

4.3. G has a finite orbit of cardinality at least 3.

Proposition 4.3. *Let G be a finitely generated periodic subgroup of quasi-rotations. If G admits a finite orbit of cardinality at least 3, then G is finite.*

Proof. Let $x_0 \in \mathbb{S}^2$ having a finite G -orbit.

As $\#\mathcal{O}_G(x_0) \geq 3$, we can write $\mathcal{O}_G(x_0) = \{x_0 = g_0(x_0), x_1 = g_1(x_0), \dots, x_n = g_n(x_0)\}$, with distinct x_i and $n \geq 2$.

We first claim that if G is not finite, then $x_0 \in P_G$.

Indeed, as G is not finite, there exists $f \notin \{Id = g_0, g_1, \dots, g_n\}$. Since $f(x_0) \in \mathcal{O}_G(x_0)$, there exists i such that $f(x_0) = g_i(x_0)$, and then x_0 is fixed by $g_i^{-1}f$.

We secondly prove that $\text{St}_G(x_0)$ is a finite group.

If $\text{St}_G(x_0)$ is not a finite group, we write $\text{St}_G(x_0) = \{f_n, n \in \mathbb{N}\}$, where $f_n \neq f_m$ if $n \neq m$.

As $\mathcal{O}_G(x_0)$ is finite, there are infinitely many $f_{s_n}, n \in \mathbb{N}$ such that $f_{s_n}(x_1) = f_{s_m}(x_1)$, for any n, m . Then $f_{s_0}^{-1} \circ f_{s_n}(x_1) = x_1$ and so $F_n = f_{s_0}^{-1} \circ f_{s_n}$ fixes x_0 and x_1 .

Analogously, there exist infinitely many $F_{k_n}, n \in \mathbb{N}$ such that $F_{k_n}(x_2) = F_{k_m}(x_2)$, for any n, m . Then $F_{k_0}^{-1} \circ F_{k_n}(x_2) = x_2$ and so $F_{k_0}^{-1} \circ F_{k_n}$ fixes x_0, x_1 and x_2 .

Consequently, $F_{k_0}^{-1} \circ F_{k_n} = Id$, since a non trivial quasi-rotation has exactly two fixed points. Finally, $F_{k_0} = F_{k_n}$, for any $n \in \mathbb{N}$. This is a contradiction. Hence $\text{St}_G(x_0)$ is finite.

Finally, we conclude that G is finite, by proving that $G = \bigcup_{i=0}^n g_i(\text{St}_G(x_0))$: let $g \in G$, as $g(x_0) \in \mathcal{O}_G(x_0)$, there exists i such that $g(x_0) = g_i(x_0)$. Hence $g_i^{-1}g \in \text{St}_G(x_0)$ and then $g \in g_i(\text{St}_G(x_0))$. \square

5. BURNSIDE PROBLEM FOR GROUPS OF HOMEOMORPHISMS OF THE REMAIN SURFACES.

In this section we prove that a finitely generated periodic group of homeomorphism of the closed disk is finite. We also prove Corollary 1 and Theorem 2.

5.1. Burnside problem for groups of homeomorphisms of the closed 2-disk.

Let Γ be a finitely generated periodic subgroup of homeomorphisms of \mathbb{D}^2 and let $C = \partial\mathbb{D}^2$. As C is invariant by Γ , according to the positive answer to the Burnside problem on the circle, Γ acts as an abelian and finite group on C and this action is faithful since any periodic homeomorphism on \mathbb{D}^2 whose restriction to C is the identity is the identity (see [3]). As a consequence, Γ is an abelian and finite group.

5.2. Burnside problem for groups of homeomorphisms of the closed annulus (Corollary 1).

First Proof using Burnside Problem on \mathbb{T}^2 .

Let Γ be a finitely generated periodic subgroup of $\text{Homeo}(\mathbb{A}^2)$. We can form \mathbb{T}^2 the double of \mathbb{A}^2 by gluing two copies of \mathbb{A}^2 : A_1 and A_2 along their boundary.

Let $g \in \Gamma$, we denote by g_i its corresponding map on A_i . We define the double \tilde{g} of g on \mathbb{T}^2 by $\tilde{g} = g_i(x)$ if $x \in A_i$. By construction \tilde{g} is a finite order (same order as g) homeomorphism that preserves the gluing boundary C and the induced action of Γ ($g \mapsto \tilde{g}$) is faithful.

Therefore Γ acts on C as a finitely generated periodic group and its subgroup preserving each boundary component is of index at most 2. According to the positive answer to the Burnside problem on the circle, this subgroup acts on each component as a finite group and the same holds for Γ . We conclude that Γ acts on \mathbb{T}^2 with a finite orbit. In particular Γ acts faithfully and preserving a probability measure on \mathbb{T}^2 , the main theorem of [8] implies that Γ is finite.

Second Proof. Let $g \in \Gamma$. Any finite order homeomorphism on the closed annulus is an isometry for some flat Riemannian metric (see [5] and [17]). Let Γ_0 be the subgroup of Γ consisting in homeomorphisms that preserve any connected component of the boundary. It is of finite index, so for proving that Γ is finite it suffices to prove that its subgroup Γ_0 , is finite. Let C_1 and C_2 be the connected components of the boundary. As C_i is invariant by Γ_0 , Γ_0 acts as an abelian and finite group on C_i and this action is faithful, since an orientation preserving isometry on closed annulus is a rotation about the central axis, if it is equal to identity on $C_1 \cup C_2$, then it is the identity on the annulus.

The forthcoming subsections provide the proof of Theorem 2.

5.3. Burnside problem for groups of homeomorphisms of a closed orientable surface S of genus g greater than one.

Let Γ be a finitely generated periodic subgroup of homeomorphisms of S .

Any finite order homeomorphism on the S is an isometry for some Riemannian metric of constant curvature equal to -1 (see [5] and [17]).

Let $\phi : \Gamma \rightarrow \text{GL}(2g, \mathbb{Z})$ be the homology morphism.

The image $\phi(\Gamma)$ is a finitely generated periodic subgroup of $\text{GL}(2g, \mathbb{Z})$, hence it is finite, according to Schur's theorem.

We will use a strong result: there is no torsion element in the Torelli's group, that is, any homeomorphism homologous to identity whose isotopic class is periodic is isotopic to identity (see, for example, Theorem 6.12 of [6]).

Then the kernel of ϕ consists in isometries isotopic to identity. But, the only isometry isotopic to identity that preserves orientation is the identity (see [6] proof of Proposition 7.7). Since the subgroup of isometries isotopic to identity that preserve orientation is of finite index in the group of isometries isotopic to identity, it follows that the kernel of ϕ is also finite. As a consequence, Γ is finite.

5.4. Burnside problem for groups of homeomorphisms of a compact orientable surface S whose boundary has more than three circles.

Let Γ be a finitely generated periodic subgroup of homeomorphisms of S . Γ preserves the boundary of S , that is, an union of circles.

We can form the double of S , \widehat{S} , by gluing two copies of S : S_1 and S_2 along their respective boundaries γ_1 and γ_2 .

Let $g \in \Gamma$, we denote by g_i its corresponding map on S_i . We define the double \tilde{g} of g on \widehat{S} by $\tilde{g} = g_i(x)$ if $x \in S_i$. By construction \tilde{g} is a finite order (same order as g) homeomorphism that preserves the gluing boundary C and the induced action of Γ ($g \mapsto \tilde{g}$) is faithful on the compact surface \widehat{S} of genus greater than one. By last subsection we have that Γ is finite.

6. PROOF OF THEOREM 3 AND COROLLARY 5.

Proof of Theorem 3.

1. Let Γ be a finitely generated periodic group of diffeomorphisms of a compact manifold M of dimension n . Suppose that Γ acts on M with a finite orbit.

Claim 1. *There exists $x_0 \in M$ such that $\Gamma_0 = St_\Gamma(x_0)$ is a finite index, finitely generated periodic subgroup of Γ .*

Proof. Let $\mathcal{O}_{x_0} = \{x_0, x_1, \dots, x_m\}$ be a finite Γ -orbit. For $i \in \{0, \dots, m\}$, we write $x_i = g_i(x_0)$, where $g_i \in \Gamma$ and $g_0 = Id_M$.

Let $g \in \Gamma$, as $g(x_0) \in \mathcal{O}_{x_0}$ there exists some $i \in \{0, \dots, m\}$ such that $g(x_0) = g_i(x_0)$. Therefore $g_i^{-1}g(x_0) = x_0$ that is $g \in g_i(St_\Gamma(x_0))$.

In conclusion, $G = \bigcup_{i=0}^m g_i(St_\Gamma(x_0))$, meaning that $St_\Gamma(x_0)$ has finite index in G ; according to Schreier's Lemma it is finitely generated. \square

Remark 3. *As a consequence of this claim, for proving that Γ is finite it suffices to prove that its subgroup Γ_0 , acting with a global fixed point on M , is finite. This is the purpose of the next proposition.*

Proposition 6.1. *Let Γ_0 be finitely generated periodic group of diffeomorphisms of M . If Γ_0 acts on M with a global fixed point then Γ_0 is finite*

Proof. Consider the map $\mathbf{D} : \Gamma_0 \rightarrow GL(n, \mathbb{R})$, where $D(g) = Dg(x_0)$ is the differential map of g at x_0 (after identification of $T_{x_0}M$ to \mathbb{R}^n).

It is easy to see that the map D is a morphism and it is faithful. Indeed, let $g \in \Gamma_0$ such that $D(g) = Id$. We have already noted that g is an isometry for some Riemannian metric (m_g) on M . Since an isometry is uniquely determined by its value and its differential at a single point, we get that $g = Id_M$.

Then Γ_0 is isomorphic to its image $D(\Gamma_0)$ which is a finitely generated periodic subgroup of $GL(n, \mathbb{R})$, hence finite, according to Schur's theorem. This concludes the case where Γ acts with a finite orbit on M

2. Let Γ be a finitely generated periodic group of diffeomorphisms of a compact manifold M . Suppose that Γ preserves a finite union of circles on M .

By an analogous argument to Claim 1, it suffices to prove that a finitely generated periodic subgroup of diffeomorphisms of M that preserves a circle is finite. According to the positive answer to the Burnside problem on the circle, Γ admits a finite orbit (on its invariant circle) so by part **1** it is finite. \square

Proof of Corollary 5.

Let f be a non trivial central element of Γ , a finitely generated periodic subgroup of $Diff^1(\mathbb{S}^2)$. As f commutes with any g in Γ , the set $Fix f$ consisting of its fixed points is G -invariant. The classification of finite order homeomorphisms of \mathbb{S}^2 indicates that $Fix f$ is either finite or a circle. Hence, Theorem 3 implies that Γ is finite. \square

7. GROUPS OF EVEN BOUNDED ORDERS

The aim of this section is proving Corollary 3.

Let G be a finitely generated periodic group of spherical diffeomorphisms with even bounded orders. The subgroup of orientation preserving elements of G is a group with even bounded orders of index at most 2 in G , then we can suppose that G consists in C^1 quasi-rotations, in particular any non trivial element of G has exactly two fixed points.

According to the classification of the finite subgroups of $Diff_+^1(\mathbb{S}^2)$ and the fact that alternating groups A_4 , A_5 and symmetric group S_4 contain elements of order 3, a finite group with even orders of orientation preserving spherical diffeomorphisms is either

- (1) a cyclic group $\mathbb{Z}/m\mathbb{Z}$ where $m = 2p$, $p \in \mathbb{N}$ or
- (2) a diedral group $\mathbb{D}_m = \langle \sigma, \tau \mid \sigma^2 = \tau^m = 1, \sigma\tau\sigma = \tau^{-1} \rangle$
 $= \langle \sigma, \sigma' \mid \sigma^2 = (\sigma\sigma')^2 = (\sigma\sigma')^m = 1 \rangle$, $m = 2p$, $p \in \mathbb{N}$.

Note that a group with even orders always contains involutions (elements of order 2) and let us denote $Inv(G) = \{\sigma \in G \setminus Id : \sigma^2 = 1\}$ and $Z(\sigma)$ the centralizer of σ in G , that is $Z(\sigma_0) = \{f \in G : f\sigma_0 = \sigma_0f\}$.

Lemma 7.1.

Let G be a finitely generated periodic group of orientation preserving spherical diffeomorphisms with even bounded orders.

- (1) *Let $\sigma_0 \in Inv(G)$, the set $Z(\sigma_0) \cap Inv(G)$ is finite,*
- (2) *$Inv(G)$ is finite.*

Proof. Let us write $Z(\sigma_0) \cap Inv(G) = \{i_n, n \in \mathbb{N}\}$, where $i_0 = \sigma_0$.

Fix $n \in \mathbb{N}$, the group G_n generated by i_0, \dots, i_n is finitely generated and fixes the set $Fix(i_0)$ (since i_k commutes with i_0), then it is finite, by Theorem 1. Therefore this group is either a cyclic group or a diedral group and $\{Id\} \subset G_0 = \langle i_0 \rangle \subset \dots \subset G_n \subset G_{n+1} \dots$.

This sequence stabilizes at some rank n_0 , since G is of bounded orders. That is, for all $n \geq n_0$ one has $G_n = G_{n_0}$, hence $i_n \in G_{n_0}$, $\forall n \geq n_0$ and therefore $Z(\sigma_0) \cap Inv(G)$ is finite.

We now prove item 2. Suppose by contradiction that there exists in G an infinite sequence of involutions $\sigma_1, \dots, \sigma_n, \dots$.

For all $n \in \mathbb{N}$, the group $\langle \sigma_0, \sigma_n \rangle$ is either cyclic or diedral. Therefore it contains an involution i_n that commutes with σ_0 and σ_n ($i_n = \sigma_0$ in the cyclic case or $i_n = (\sigma_0 \sigma_n)^{\frac{m_n}{2}}$ in the diedral case $\langle \sigma_0, \sigma_n \rangle = \mathbb{D}_{m_n}$).

Since $Z(\sigma_0) \cap \text{Inv}(G)$ is finite, we can suppose (eventually passing to an infinite subsequence of (σ_n)) that $i_n = i$, for all n .

Finally, i commutes with all σ_n , that is $\{\sigma_n, n \in \mathbb{N}\} \subset Z(i) \cap \text{Inv}(G)$ which is finite by item 1, that is a contradiction. \square

End of proof of Corollary 3.

Applying Lemma 7.1, we obtain that $\text{Inv}(G)$ and therefore $\text{Fix}(\text{Inv}(G)) = \{x \in \mathbb{S}^2 : \sigma(x) = x, \text{ for some } \sigma \in \text{Inv}(G)\}$ are finite sets.

As $\text{Fix}(f\sigma f^{-1}) = f(\text{Fix}(\sigma))$ and $f\sigma f^{-1} \in \text{Inv}(G)$ for all $f \in G$, the finite set $\text{Fix}(\text{Inv}(G))$ is non empty and G -invariant. By Theorem 1, G is finite. \square

REFERENCES

- [1] Adjan S.I. and Novikov P.S. *On infinite periodic groups I, II, III.* Izv. Akad. Nauk SSSR. Ser. Mat. 32 (1968), 212-244; 251-524; 709-731.
- [2] Burnside W. *On unsettled question in the theory of discontinuous groups.* Quart. J. Math. 33 (1902), 230-238.
- [3] Constantin A. and Kolev B. *The theorem of Kerekjerto on periodic homeomorphisms of the disc and the sphere.* Enseign. Math. (2), vol 40, (1994), no. 3-4, 193-204.
- [4] Eilenberg S. *Sur les transformations périodiques de la surface de la sphère.* Fund. Math. 22 (1934), 28-41.
- [5] Epstein D. *Pointwise periodic homeomorphisms.* Proc. London Math. Soc. 42 (1981), 415-460.
- [6] Farb B. and Margalit D. *A Primer on Mapping Class Groups.* Princeton University Press (2011)
- [7] Golod E. S. *On nil algebras and finitely residual groups.* Izv. Akad. Nauk SSSR. Ser. Mat. 1975, (1964), 273-276.
- [8] Guelman N. and Liousse I. *Burnside problem for measure preserving groups and for 2-groups of toral homeomorphisms.* Geometriae Dedicata 168 (2014), no. 1, 387-396.
- [9] Ivanov S. *The free Burnside groups of sufficiently large exponents.* Internat. J. Algebra Comput., no. 4, (1994), 1-308.
- [10] Von Kerekjerto B. *Über die periodischen transformationen der Kreisscheibe und der Kugelfläche.* Math. Ann. 80 (1919-1920), 36-38.
- [11] Lang S. $\text{SL}_2(\mathbb{R})$. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, (1975).
- [12] Lysenok I. *Infinite Burnside groups of even period.* Math. Ross. Izv., no. 60 (1996), 3-224.
- [13] Meeks W. and Scott P. *Finite group actions on 3-manifolds.* Invent. Math. 86 (1986), no. 2, 287-346.
- [14] Navas A. *Groups of circle diffeomorphisms.* Chicago lectures in mathematic series (2011).
- [15] Ol'Shanskii A. *On the Novikov-Adian theorem.* Math. USSR Sb, no. 118 (1982), 203-235.
- [16] Rebelo J. and Silva A. *On the Burnside problem in $\text{Diff}(M)$.* Discrete Contin. Dyn. Syst. 17 (2007), no. 2, 423-439.
- [17] Stillwell J. *Geometry of surfaces* Springer-Verlag, New York, 1992
- [18] Schur I. *Über Gruppen linearer Substitutionen mit Koeffizienten aus einem algebraischen Zahlkörper.* (German) Math. Ann. 71 (1911), no. 3, 355-367.

Nancy Guelman IMERL, FACULTAD DE INGENIERÍA, UNIVERSIDAD DE LA REPÚBLICA, C.C. 30, MONTEVIDEO, URUGUAY. nguelman@fing.edu.uy.

Isabelle Liousse, UMR CNRS 8524, UNIVERSITÉ DE LILLE1, 59655 VILLENEUVE D'ASCQ CÉDEX, FRANCE. liousse@math.univ-lille1.fr.